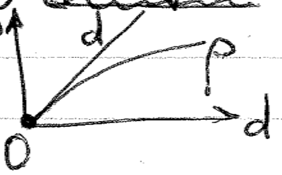
Solutions.

Axiom(1):

①(a): $\varrho(x,y) = \ln(1+d(x,y)) \geqslant 0$ $(\forall d(x,y) \geqslant 0)$; & $\rho = 0 \Leftrightarrow d = 0$ $(x = y)$.

Ax.(2): symm $\Leftarrow$ (d was symm.); • Ax(3): $\Delta\neq$: $\forall x,y,z \in \mathfrak{X}$

$$\ln(1+d(x,z)) \ (\leqslant ?) \ \ln(1+d(x,y)) + \ln(1+d(y,z))$$



Lemma: exp is monotonic on $\mathbb{R}$; exp(? inequality):

$$1 + d(x,z) \ (\leqslant ?) \ (1+d(x,y))\cdot(1+d(y,z)) = 1 + d(x,y) + d(y,z) + d(x,y)\cdot d(y,z).$$

$\Delta\neq(d)$: $d(x,z) \leqslant d(x,y) + d(y,z) \overset{!}{\leqslant} d(x,y) + d(y,z) + \underbrace{d(x,y)\cdot d(y,z)}_{\geqslant 0}$;

Reading inequalities upwards, $\Rightarrow$ Q.E.D.

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①(b): NB: Continuous (w.r.t. $\mathbb{E}^1 \rightleftarrows \mathbb{E}^1$) recalculation $d \rightleftarrows \varrho$ of the values of metrics makes $id: (\mathfrak{X}, d) \to (\mathfrak{X}, \varrho)$ and id^{-1} uniformly continuous (w.r.t. $\forall x,y \in \mathfrak{X}$ in $\varrho(x,y)$ or in $d(x,y)$.)

• $(\mathfrak{X}, d) \to (\mathfrak{X}, \varrho)$: Note $\varrho = \ln(1+d) \leqslant d$ $(\forall x,y \in \mathfrak{X})$ $\Rightarrow \left\{ \begin{array}{c} d < \varepsilon \\ \Downarrow \\ \varrho < \varepsilon \end{array} \right.$.

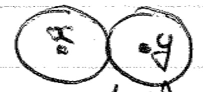
$\forall \varepsilon > 0$, take $\delta(\varepsilon \mid \forall x,y) := \varepsilon \Rightarrow d(x,y) < \delta = \varepsilon \Rightarrow \varrho(x,y) < \varepsilon$.

Take $\forall$ Cauchy $\{x_n\} \subseteq \mathfrak{X}$ wrt d: $\exists N \in \mathbb{N} \mid \forall m > n \geqslant N \Rightarrow d(x_m, x_n) < \delta = \varepsilon$ $\Rightarrow \varrho(x_m, x_n) < \varepsilon \}$ $\Rightarrow$ $id(\{x_n\}) =: \{y_n\}$ is Cauchy in $(\mathfrak{X}, \varrho)$.

Now, under the assumption that $(\mathfrak{X}, \rho)$ is complete $\Rightarrow$ Cauchy $\{y_n\}$ converges in $\mathfrak{X}$ to $y \in \mathfrak{X}$ (wrt ϱ). By [uniform] continuity of id^{-1} (wrt $(d \& \varrho)$), $\{x_n\} = \{id^{-1}(y_n)\}$ converges to $id^{-1}(y) = y$ in $(\mathfrak{X}, d)$.

• Likewise, $(\mathfrak{X}, d) \leftarrow (\mathfrak{X}, \rho)$: To have $d < E$, take $\varrho = \ln(1+d) < D := \ln(1+E)$. $(\Rightarrow d < E)$ because $\rho(d)$ is monotonic $\Leftrightarrow d(\rho)$ is monotonic.

Now repeating the above argument with $\varrho \rightleftarrows d$ and $\varepsilon \rightleftarrows E$, $\delta \rightleftarrows D$, $\Rightarrow$ $\forall$ Cauchy wrt ρ is Cauchy wrt d, hence if $(\mathfrak{X}, d)$ complete $\Rightarrow$ so is $(\mathfrak{X}, \rho)$ $\leftarrow$ also complete.



②(a): NB: Every metric space is Hausdorff: $\varepsilon := \frac{1}{2}\cdot dist(x,y)$ $\Leftarrow \Delta\neq$.

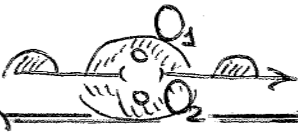
Put $\mathcal{U}_x := \bigcap_{\text{(all such sets)}} \{\text{open in Hausdorff } \mathfrak{X}, ⑧ \ni x\}$.

• {Every such set $\ni x$} $\Rightarrow$ $\mathcal{U}_x \ni x \Rightarrow \{x\} \subseteq \mathcal{U}_x$.

• $\forall y \neq x$, by Hausdorff, $\{\exists\, \mathcal{U}'_x \in \mathfrak{M}_{\mathfrak{X}} \mid \mathcal{U}'_x \ni x;\ \exists\, V'_y \in \mathfrak{M}_{\mathfrak{X}} \mid V'_y \ni y\} \mid \mathcal{U}'_x \cap V'_y = \varnothing$. That is, $\mathcal{U}'_x$ is in the list in $\bigcap$, yet now $y \notin \mathcal{U}'_x$.

$\Rightarrow \{\forall y \neq x,\ y \notin \mathcal{U}_x\}$. $\Rightarrow$ $\mathcal{U}_x = \{x\}$.

②(b): Ex: Thick line.



[DEF. in L8.]

③ Fix $\forall x_0 \in A$; $\forall y \in A \Rightarrow \exists$ connected $B(x_0,y) \subseteq A$, $B(x_0,y) \ni y, \ni x_0$

Now $A = \bigcup_{y\in A} B(x_0,y)$ and $\bigcap_{y\in A} B(x_0,y) \supseteq \{x_0\} \neq \emptyset$.

• Suppose "A not connected: "$\exists$" $f: A \xrightarrow[\text{onto}]{\text{cont}} (\{0,1\}, d_0)$"; without loss of generality $x_0 \in f^{-1}(0)$; $f^{-1}(1) \neq \emptyset \Rightarrow \exists\, y \in f^{-1}(y) \subseteq A$.

$\Rightarrow$ "Partition" $A = \underset{x_0}{f^{-1}(0)} \sqcup \underset{y}{f^{-1}(1)}$. (open in A) (open in A). Recall $B(x_0,y) \subseteq A$.

$\Rightarrow \left\{\begin{array}{l} B(x_0,y) \cap f^{-1}(0) \ni x_0, \text{ open in } A \\ B(x_0,y) \cap f^{-1}(1) \ni y, \text{ open in } A \end{array}\right\} \Rightarrow$ "partition of $B(x_0,y)$". Contradiction.

Σ: All of A is in the connected component $f^{-1}(0)$ of $x_0 \Rightarrow A$ connected.

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④ $\equiv$: If $\mathfrak{X}$ is NOT sequentially compact $\Rightarrow \exists$ continuous unbound $f: \mathfrak{X} \to \mathbb{E}^1$.

Proof. • Use [Cor. 14.13] from L12/book:

$\{\mathrm{In}(\mathfrak{X}, d_{\mathfrak{X}}),$ if $\{\exists \{x_n\} \subseteq \mathfrak{X} \mid \nexists$ any convergent subsequence$\} \Rightarrow$

$\Rightarrow \{\forall x \in \mathfrak{X}, \exists \varepsilon(x) > 0 \mid$ disk $B^{d_{\mathfrak{X}}}_{\varepsilon(x)}(x)$ contains x_n for $\#n < \infty.\}$

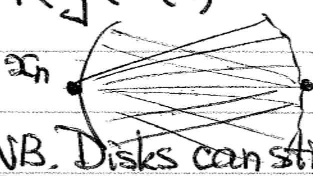
1] $\mathfrak{X}$ not seq. compact $\Rightarrow \exists \{x_n\} \subseteq \mathfrak{X}$ without converg. (in $\mathfrak{X}$) subseq.

2] Apply Cor. 14.13 to points x_n of $\{x_n \mid n \in \mathbb{N}\} \subseteq \mathfrak{X}$: $\forall n \in \mathbb{N}$,

$\exists \varepsilon(x_n) > 0 \mid B^{d_{\mathfrak{X}}}_{\varepsilon(x_n)}(x_n) \ni x_{k_1}, \dots, x_{k_{r(n)<\infty}}$;

3] $\forall$ finite set of numbers $d_{\mathfrak{X}}(x_n, x_{k_j})$, $1 \leq j \leq r(n)$ contains a minimal such number $=: \tilde{\varepsilon}(x_n)$;

$\Rightarrow B^{d_{\mathfrak{X}}}_{\tilde{\varepsilon}(x_n)>0}(x_n) \not\ni x_k \mid \forall k \neq n$.

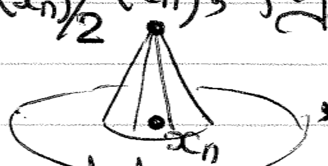


4] Shrink all $\tilde{\varepsilon}(x_n)$ twice. NB. Disks can still overlap

$\Rightarrow$ By $\Delta\neq$, $\forall y \in \mathfrak{X}$ now belongs to at most one $B_{\tilde{\varepsilon}(x_n)}(x_n)$ [or to none].

5] Define $f(y)$: $\forall y \notin B_{\tilde{\varepsilon}(x_n)/2}(x_n)$, $f(y) := 0$

• Inside $B_{\tilde{\varepsilon}(x_n)/2}(x_n)$



f attains value n at x_n: $f(y) = n - n \cdot \dfrac{d_{\mathfrak{X}}(x_n, y)}{\tilde{\varepsilon}(x_n)/2}$;

6] f continuous on $\mathfrak{X}$, unbounded.

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⑤ Ex. $[1,+\infty) \subsetneq \mathbb{E}^1$ $\}$ $\leftarrow$ Closed in $\mathbb{E}^1$ complete $\Rightarrow [1,+\infty)$ is complete.

• $f(x) := x + \frac{1}{x} > x \Rightarrow \forall x \in [1,+\infty)$, $f(x) \neq x$. $\}$ $\nexists$ fixed point.

• $f'(x) = 1 - \underbrace{1/x^2}$ $\} \Rightarrow \forall x,y \in [1,+\infty)$, $|f(x) - f(y)| = |(y-x) \cdot f'(\xi)| =$

$= |y-x| \cdot |f'(\xi)| < |y-x|$ $(\exists\, \xi \in [x,y])$: $d(f(x), f(y)) < d(x,y)$

$(\forall x \neq y)$ $\qquad \forall x \neq y$.